Math 245B Lecture 21 Notes

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1 L^∞ Spaces and Duality of L^p Spaces

1.1 Properties of L^∞ spaces

Theorem 1.1. *L^∞ has the following properties:*

1. *For all measurable f, g, $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.*
2. *$\|\cdot\|_\infty$ is a norm.*
3. *L^∞ is complete.*
4. *$f_n \to f$ in L^∞ iff there exists $E \in \mathcal{M}$ with $\mu(E^c) = 0$ such that $f_n|_E \to f|_E$ uniformly.*
5. *The set of simple functions (not necessarily integrable) is dense in L^∞.*

Remark 1.1. If $\mu \not\equiv 0$, we can write

$$\|f\|_\infty := \inf\{a \geq 0 : \mu(\{|f| > a\}) = 0\} = \sup\{b : \mu(\{|f| > b\})\}.$$

The infimum in the definition is attained, but the supremum may not be. Let $a = \|f\|_\infty$. Let $a_n \downarrow 0$ and $\mu\{|f| > a_n\} = 0$. Now

$$\mu(\{|f| > a\}) = \mu\left(\bigcup_n \{|f| > a_n\}\right) = 0.$$

If $a = \|f\|_\infty$, then $\mu(\{|f| > a\}) = 0$. Define

$$g = \begin{cases} f & |f| \leq a \\ 0 & |f| > a. \end{cases}$$

Now $g = f$ a.e., and $\|g\|_u = \|f\|_\infty$.

Remark 1.2. If $\mu \ll \nu$ and $\nu \ll \mu$, then $L^\infty(\mu) = L^\infty(\nu)$.

Remark 1.3. On $\mathbb{R}^n$, the set of continuous functions with bounded support is not dense in L^∞. Indeed, $C[0,1] \subseteq L^\infty([0,1], m)$. Then if f is continuous, $\|f\|_\infty = \|f\|_u$.

1.2 Duality of L^p spaces

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 \le p < \infty$. Let q be such that $p^{-1} + q^{-1} = 1$. So $1 < q \le \infty$.

Let $g \in L^q$. Then for all $f \in L^p$,

$$\left| \int fg \, d\mu \right| \le \int |fg| \, d\mu \le \|f\|_p \|g\|_q$$

by Hölder's inequality. So if we define $\varphi_g : L^p \to \mathbb{C}$ sendsing $f \mapsto f \mapsto \int fg \, d\mu$, then $\varphi_g \in (L^p)^*$, and $\|\varphi g\|_{(L^p)^*} \le \|g\|_{L^q}$.

Theorem 1.2 (Riesz representation[1])**.** *If $1 < p < \infty$, then $L^q \to (L^p)^*$ sending $g \mapsto \varphi_g$ is an isometric isomorphism. The same is true if $p = 1$, provided μ is σ-finite.*

Remark 1.4. When $p = \infty$, $q = 1$. For basically any nontrivial measure, $(L^\infty)^*$ is actually much bigger than L^1.

In this lecture, henceforth, μ is a finite measure. The extension to σ-finite measures is obtained by splitting up the space into countably many pieces and applying these results to each piece.

Proposition 1.1. *If $g \in L^q$, then $\|\varphi_g\| = \|g\|_q$.*

Proof. We have already shown one of the inequalities. If $q < \infty$, (i.e. $p > 1$), then let

$$f := \frac{|g|^{q-1}\overline{\operatorname{sgn}(g)}}{\|g\|_q^{q-1}}.$$

Then, because $p(q-1) = q$, we have

$$|f|^p = \frac{|g|^q}{\|g\|_q^q},$$

so $\int |f|^p = 1$. But now

$$fg = \frac{|g|^q}{\|g\|_q^{q-1}} |g| \operatorname{sgn}(g) = \frac{|g|^q}{\|g\|_q^q},$$

so $\int fg = \|g\|$. That is, $\|\varphi_g\| = \|\varphi_g\| \|f\|_p \ge \int fg = \|g\|_q$.

If $q = \infty$, i.e. g is essentially bounded, let $\varepsilon > 0$. Then $0 < \mu(\{|g| \ge \|g\|_\infty - \varepsilon\}) < \infty$. Now let

$$f = \mathbb{1}_{\{|g| \ge \|g\|_\infty - \varepsilon\}} \overline{\operatorname{sgn}(g)}.$$

[1] There are many theorems called the Riesz representation theorem, all by the same person. Riesz was a busy guy.

Then $f \in L^1$, and $\|f\| = \mu(\{|g| \geq \|g\|_\infty - \varepsilon\})$. Also,

$$\int fg \, d\mu = \int_{\{|g| \geq \|g\|_\infty - \varepsilon\}} |g| \overline{\operatorname{sgn}(g)} \, d\mu \geq (\|g\| - \varepsilon)\|f\|,$$

so $\|\varphi_g\| \geq \|g\|_\infty - \varepsilon$ for all $\varepsilon > 0$. $\square$

Remark 1.5. If μ is finite, $L^q \subseteq L^1$ for all $q \geq 1$.

Proposition 1.2. *Let $g \in L^1$, and let Σ be the set of simple functions on X. Then*

$$\|g\|_q = \sup\left\{\left|\int fg\right| : f \in \Sigma, \|f\|_p \leq 1\right\}.$$

Proof. We already have that $\|g\|_q$ is at least as much as the right hand side, so it is enough to show the reverse.

Step 1: $|fg| \leq \text{RHS}$ for bounded, measurable functions: For all such f, $\|f\|_p \leq 1$. Given this f, there exist simple functions $f_n \to f$ pointwise such that $|f_n| \uparrow |f|$. In particular, $f_n \in \Sigma$, and $\|f_n\|_p \leq 1$. Also, $f_n g \to fg$ a.e., and $|f_n g| \leq |fg|$ for all n. Then $fg \in L^1$ because $\|fg\|_1 \leq \|f\|_\infty \|g\|_1$. So, by the DCT, $|\int f_n g| \to |\int fg|$; since the sequence terms are all bounded by the RHS of the inequality we want to show, so is the limit.

Step 2: $\|g\|_q \leq \text{RHS}$. Assume $q < \infty$. There exist simple functions $\varphi_n \to g$ pointwise such that $|\varphi_n| \uparrow |g|$. By the previous proposition, there exist simple functions f_n such that $\|\varphi_n\|_q = |\int f_n \varphi_n|$. Then, by the monotone convergence theorem,

$$\|g\|_q = \lim_n \|\varphi_n\|_q = \lim_n \left|\int f_n \varphi_n\right| \leq \lim_n \int |f_n||\varphi_n| \leq \lim_n \int |f_n||g| \leq \text{RHS}. \qquad \square$$

We have shown so far that

$$\|g\|_q = \sup\left\{\left|\int fg\right| : f \in L^p, \|f\|_p = 1\right\} = \sup\left\{\left|\int fg\right| : f \in \Sigma, \|f\|_p = 1\right\}.$$

We will finish up the rest of the proof next time.